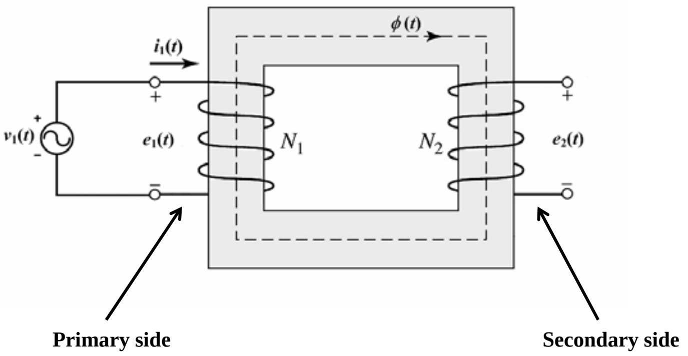
# II. Transformers

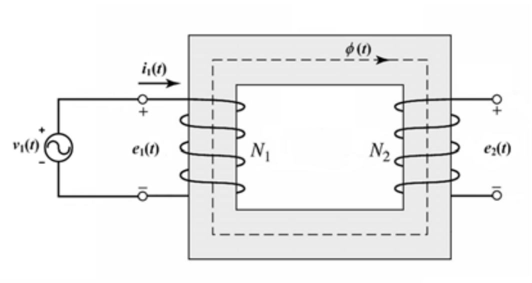
## Transformer

Transformer comprises two or more windings coupled by a common magnetic circuit (M.C.).

If the primary side is connected to an AC voltage source $\boldsymbol{v_1(t)}$, an AC flux $\boldsymbol{\phi(t)}$ will be produced in the M.C.



Ideal transformer voltage relationship

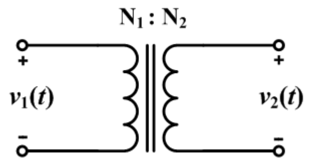



$$e_1(t) = N_1 \frac{d\phi(t)}{dt}$$

$$e_2(t) = N_2 \frac{d\phi(t)}{dt}$$

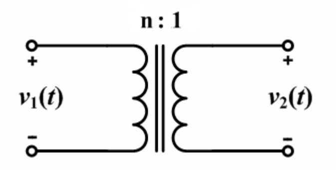
$$\boxed{\frac{e_1}{e_2} = \frac{N_1}{N_2}}$$

**Ideal transformer symbol**



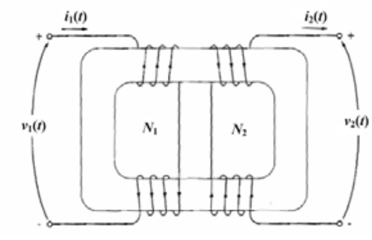

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$

Another representation

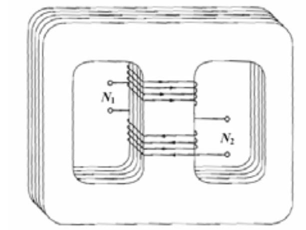



$$n = \frac{N_1}{N_2}$$

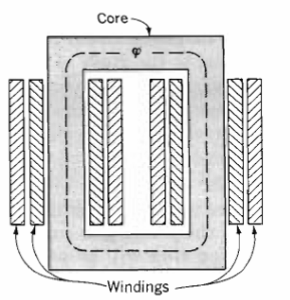
There are two common types of transformer construction:

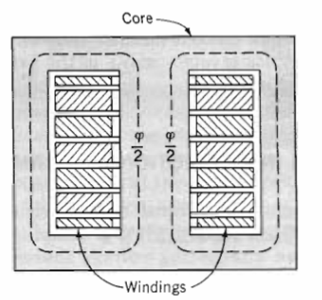



**(a) core-type**



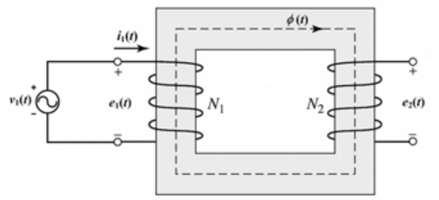

**(b) shell-type**

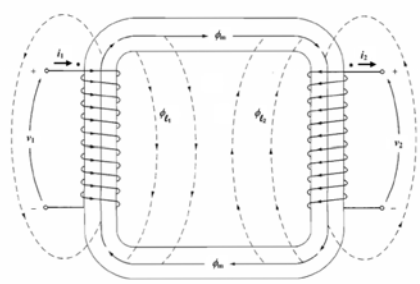


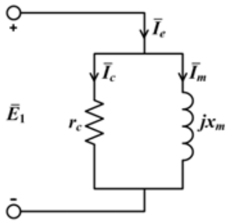

# 1. Analysis of the transformer with no-load

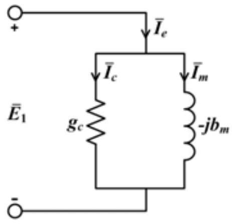
Secondary side
open-circuited:
(no sec. current)



## Modelling of Magnetic Core




$r_c$: core loss resistance
(hysteresis & eddy-current loss)

$x_m$: magnetizing reactance

$I_e$ : exciting current

$g_c$: core loss conductance

$g_c$ - $jb_m$: magnetizing admittance

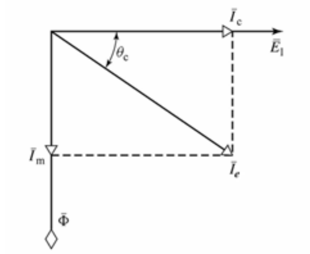
$b_m$ : magnetizing susceptance

$$g_c = \frac{1}{r_c}$$

$$-jb_m = \frac{1}{jx_m}$$

$$\bar{I}_e = \bar{I}_c + \bar{I}_m$$

**Phasor diagram**




$\bar{I}_m$ lags $\bar{E}_1$ by 90º

$\bar{I}_c$ in phase with $\bar{E}_1$

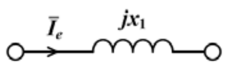
## Modelling of Leakage Flux

Let us express the voltage drop due to leakage flux in the primary winding below

$$v_{\ell_1} = \frac{d\lambda_{\ell_1}}{dt} = \frac{d\lambda_{\ell_1}}{di_e}\frac{di_e}{dt} = L_{\ell_1}\frac{di_e}{dt} \qquad \text{where} \qquad \lambda_{\ell_1} = N_1\phi_{\ell_1}$$

$$v_{\ell_1} = L_{\ell_1}\frac{di_e}{dt}$$

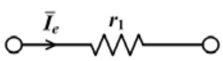
$L_{\ell_1}$ : leakage inductance of primary winding




primary leakage reactance: $x_1 = \omega L_{\ell_1}$

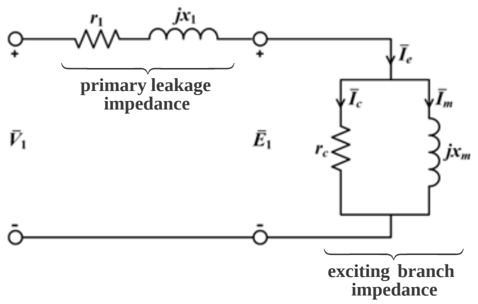
$$\omega = 2\pi f$$

## Modelling of Copper Loss




$r_1$ : resistance of primary winding

**Equivalent circuit model of primary side**




Exciting current is only a few percent of rated primary current of the transformer

# 2. Ideal Transformer Operation

- No leakage fluxes
- Negligible winding internal resistances
- B-H characteristic of the magnetic material is single-valued, and linear
  - No hysteresis loss
- Magnetic core has a very high $\mu_r$, i.e. Core reluctance is negligible.
- No copper, no core losses (Efficiency $\eta$ = 100%)
- Interwinding capacitatances are neglígle at power frequencies (50Hz, 60Hz)

# Basic Relations

1. From Faraday's Law: $e_1 = N_1\, d\phi/dt$, and
   $e_2 = N_2\, d\phi/dt$

   So,

$$\boxed{\frac{e_1}{e_2} = \frac{N_1}{N_2}}$$

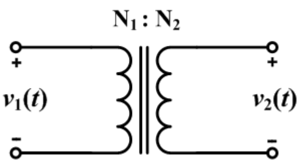
2. Since winding resistances & leakage fluxes are negligible:

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$

3. $\mathcal{F}_1 = \mathcal{F}_2$

$$\boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}}$$

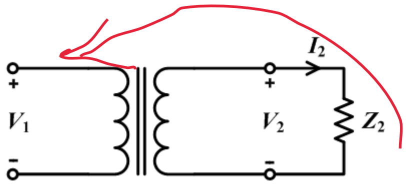
4. Ideal transformer symbol



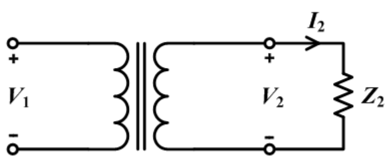
4. No power loss

Conservation of power: $v_1 i_1 = v_2 i_2$

5. Under Load




# Ideal transformer under load

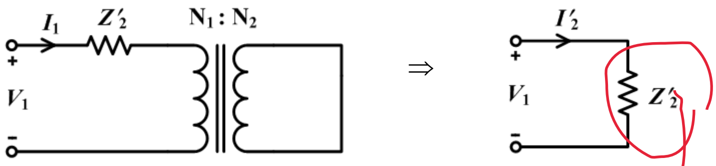



$\frac{v_1}{v_2} = \frac{N_1}{N_2}$

$\frac{i_1}{i_2} = \frac{N_2}{N_1}$

$v_1 i_1 = v_2 i_2$

$v_2 = i_2 Z_2$




$I_1 = I'_2$

Eqv. crt. referred to primary

Secondary impedance referred to primary side: $Z'_2 = \left(\frac{N_1}{N_2}\right)^2 Z_2 = n^2 Z_2$

## Terminology

$V_1,\ I_1,\ Z_1:$ actual primary quantities

$V_2,\ I_2,\ Z_2:$ actual secondary quantities

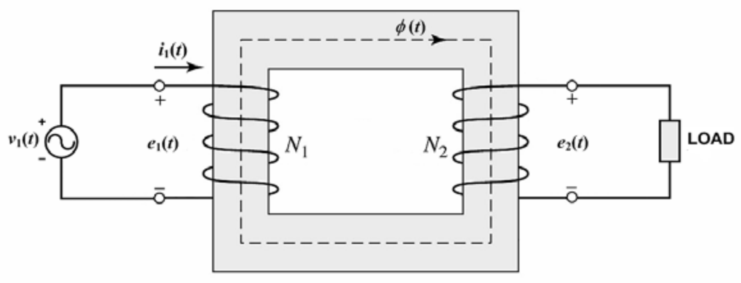
$V_1',\ I_1',\ Z_1':$ primary quantities referred to secondary side

$V_2',\ I_2',\ Z_2':$ secondary quantities referred to primary side

$$V_1' = \frac{N_2}{N_1} V_1, \quad I_1' = \frac{N_1}{N_2} I_1, \quad Z_1' = \left(\frac{N_2}{N_1}\right)^2 Z_1$$

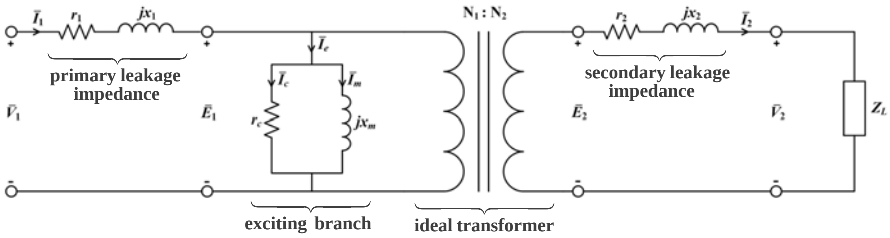
$$V_2' = \frac{N_1}{N_2} V_2, \quad I_2' = \frac{N_2}{N_1} I_2, \quad Z_2' = \left(\frac{N_1}{N_2}\right)^2 Z_2$$

## 3. Equivalent circuit representation of a practical transformer




Transformer under load

## Equivalent Circuit representation



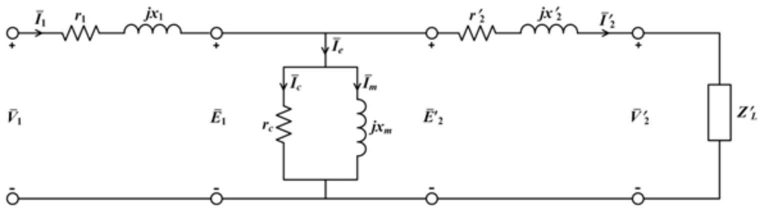

$r_1$: Primary winding internal resistance (Ω) $r_2$: Secondary winding internal resistance (Ω)

$x_1$: Primary winding leakage reactance (Ω) $x_2$: Secondary winding leakage reactance (Ω)

$r_c$: Core-loss resistance (Ω)

$x_m$: Magnetizing reactance (Ω)

## Equivalent circuit referred to primary side



$r'_2$: Secondary winding internal resistance referred to primary side

$x'_2$: Secondary winding leakage reactance referred to primary side

$I'_2$: Secondary winding current referred to primary side

$V'_2$: Secondary winding voltage referred to primary side

$Z'_L$: Load impedance referred to primary side

$$r'_2 = n^2 r_2$$

$$x'_2 = n^2 x_2$$

$$I'_2 = \frac{I_2}{n}$$

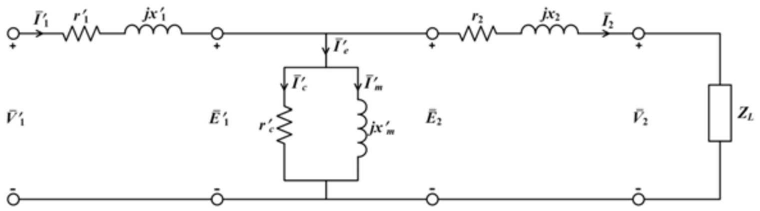
$$V'_2 = nV_2$$

$$Z'_L = n^2 Z_L$$

$$n = \frac{N_1}{N_2}$$

$$E_1 = E'_2 = nE_2$$

# Equivalent circuit referred to secondary side




$r'_1$: Primary winding internal resistance referred to secondary side

$x'_1$: Primary winding leakage reactance referred to secondary side

$I'_1$: Primary winding current referred to secondary side

$V'_1$: Primary winding voltage referred to secondary side

$r'_c$: Core-loss resistance referred to sec. side

$x'_m$: Magnetizing reactance referred to sec. side

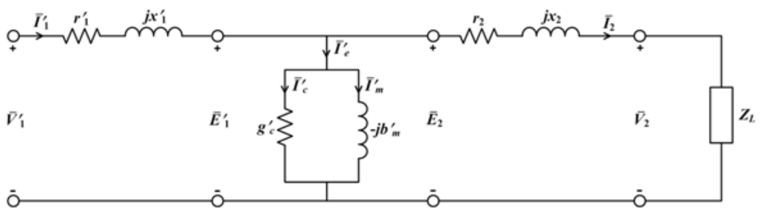
$$r'_1 = \frac{r_1}{n^2} \qquad n = \frac{N_1}{N_2}$$

$$x'_1 = \frac{x_1}{n^2} \qquad I'_1 = nI_1$$

$$r'_c = \frac{r_c}{n^2} \qquad V'_1 = \frac{V_1}{n}$$

$$x'_m = \frac{x_m}{n^2} \qquad E_2 = E'_1 = \frac{E_1}{n}$$


# Equivalent circuit referred to secondary side


$g'_c$ : Core-loss conductance referred to sec. side

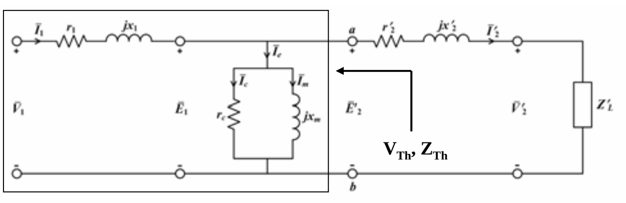
$b'_m$ : Magnetizing admittance referred to sec. side

$$n = \frac{N_1}{N_2}$$

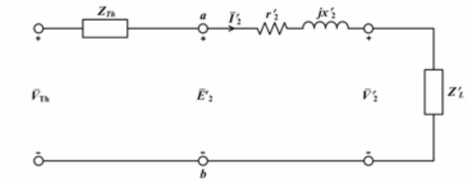
$$g'_c = \frac{1}{r'_c} \qquad -jb'_m = \frac{1}{jx'_m}$$

$$g'_c = n^2 g_c \qquad b'_m = n^2 b_m$$

# Simplification




**Let us apply Thévenin equivalent circuit to the primary side**



$$r_c \parallel jx_m = \frac{1}{g_c - jb_m}$$

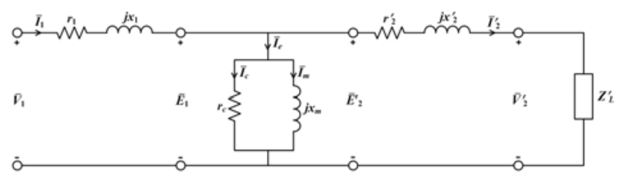
$$\overline{V}_{Th} = \overline{V}_1 \frac{r_c \parallel jx_m}{\cancel{(r_1 + jx_1)} + (r_c \parallel jx_m)} \cong \overline{V}_1 \qquad \textbf{where} \quad |r_c \parallel jx_m| >> |r_1 + jx_1|$$

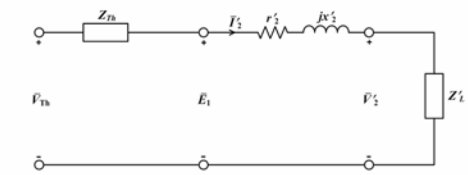
$$\boxed{\overline{V}_{Th} \cong \overline{V}_1}$$

$$Z_{Th} = (r_1 + jx_1) \parallel (r_c \parallel jx_m) \cong r_1 + jx_1$$

$$\boxed{Z_{Th} \cong r_1 + jx_1}$$

# Simplification



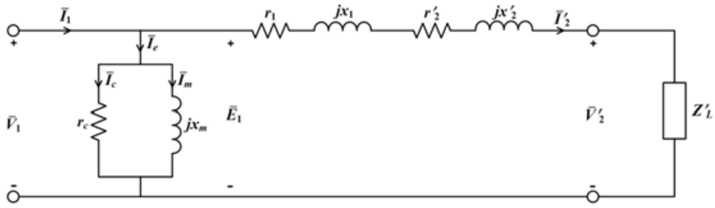


$$\overline{E}_1 = \overline{V}_{Th} - \cancel{Z_{Th}\overline{I}'_2} \cong \overline{V}_{Th} \cong \overline{V}_1 \qquad \textbf{where} \quad Z_{Th} \cong r_1 + jx_1$$

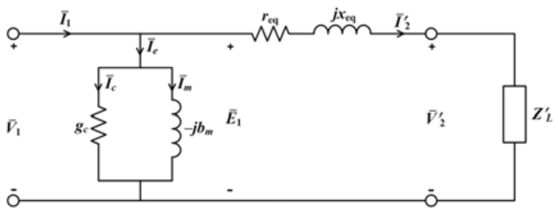
$$\boxed{\overline{E}_1 \cong \overline{V}_1}$$

Error made in approximations is at most 1% – 2% compared to actual values.

# Approximate Equivalent Circuit referred to primary side



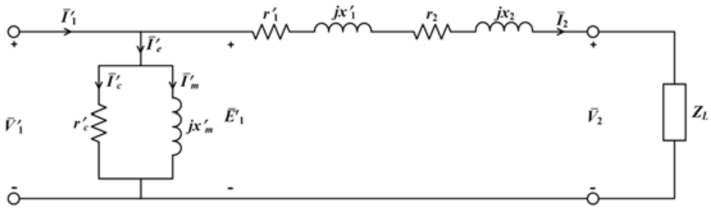
Further simplification gives us the figure below



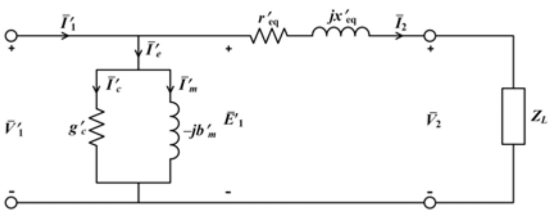
$$r_{eq} = r_1 + r_2'$$

$$x_{eq} = x_1 + x_2'$$

# Approximate Equivalent Circuit referred to secondary side



Further simplification gives us the figure below

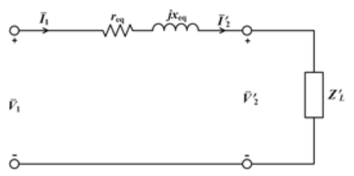


$$r_{eq}' = r_1' + r_2$$

$$x_{eq}' = x_1' + x_2$$

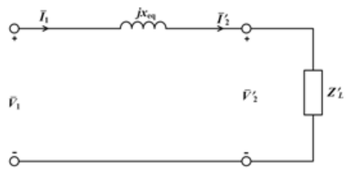
## Approximate Equivalent Circuits for Large Transformers (referred to primary side)

**(of a few 100 kVAs)**



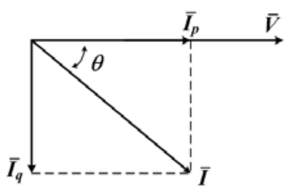
$|r_c \parallel jx_m|$ **is very large**

**(in the MVA range)**



$x_{eq} \gg r_{eq}$ **(4 - 10 times)**

# AC Power




$\theta$: **angle between voltage** $\bar{V}$ **and current** $\bar{I}$

Real Power : $P = V_{rms} I_{rms} \cos\theta$ **[W, Watts]**

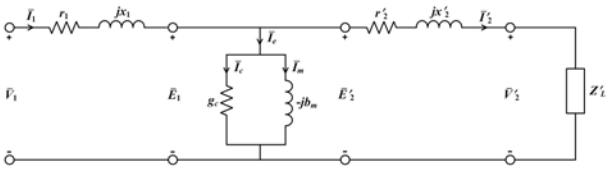
Reactive Power (Imaginary Power) : $Q = V_{rms} I_{rms} \sin\theta$ **[VAR, Volt Ampere Reactive]**

Complex Power : $S = P + jQ$

Apparent Power : $|S| = V_{rms} I_{rms}$ **[VA, Volt Ampere]**

Power Factor : $\cos\theta = \dfrac{P}{|S|}$

## Transformer Power Flow




$P_{\text{in}} = V_1\, I_1 \cos\theta_1$

$P_{\text{out}} = P_{\text{load}} = V'_2\, I'_2 \cos\theta_2$

$P_{\text{in}} = P_{\text{cu}_1} + P_{\text{core}} + P_{\text{cu}_2} + P_{\text{load}}$

$P_{\text{cu}}$ : Copper loss

$P_{\text{core}}$ : Core loss

$P_{\text{cu}_1} = I_1^2 r_1 = I_1'^2 r_1'$

$P_{\text{core}} = E_1^2 g_c = E_1'^2 g_c'$

$P_{\text{core}} \cong V_1^2 g_c$

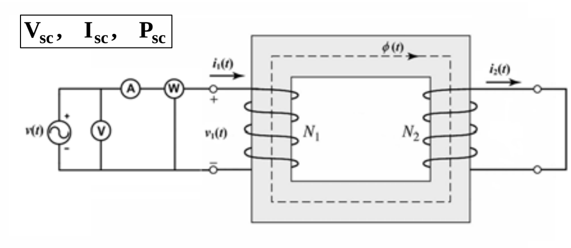
$P_{\text{cu}_2} = I_2^2 r_2 = I_2'^2 r_2'$

# 4. Short-circuit and Open-circuit Tests

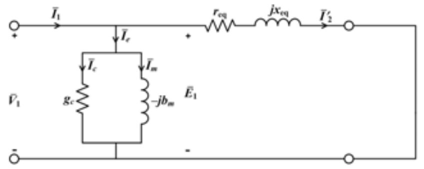
- Measure voltage (V), current (I) and power (P) in order to determine the equivalent circuit parameters of the transformer:
  - For leakage impedance parameters
    - with secondary short circuited
  - For exciting branch parameters
    - with secondary open circuited

## i. Short-circuit Test

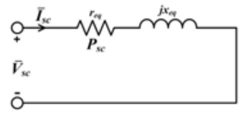
**$V_{sc}$, $I_{sc}$, $P_{sc}$**




A reduced voltage $\mathbf{V_{sc}}$ of 2% - 10 % of rated voltage is applied to allow rated primary current.

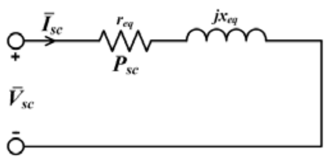



**where** $\dfrac{1}{g_c - jb_m} >> r_{eq} + jx_{eq}$




**Short circuit equivalent circuit**

## Short-circuit Test




$$r_{eq} = \frac{P_{sc}}{I_{sc}^2}$$

$$|z_{eq}| = \frac{V_{sc}}{I_{sc}}$$

**where** $z_{eq} = r_{eq} + j\,x_{eq}$

$$x_{eq} = \sqrt{|z_{eq}|^2 - r_{eq}^2}$$

$r_{eq} = r_1 + r_2'$ **where** $r_1 \cong r_2'$

$$r_1 = r_2' \cong \frac{r_{eq}}{2}$$

$x_{eq} = x_1 + x_2'$ **where** $x_1 \cong x_2'$

$$x_1 = x_2' \cong \frac{x_{eq}}{2}$$

**at 50 Hz** $r_{1_{AC}} = 1.1\, r_{1_{DC}}$

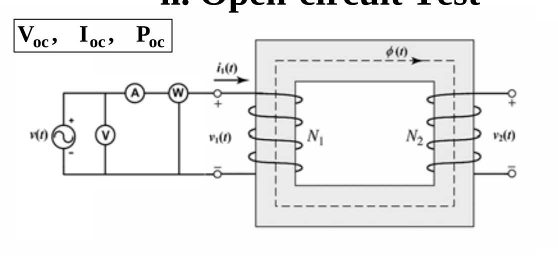
$r_{2_{AC}} = 1.1\, r_{2_{DC}}$

**Form factor**

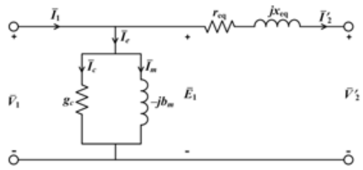
**Measured DC resistance**

# ii. Open-circuit Test

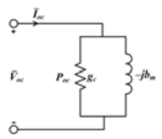
**$V_{oc}$, $I_{oc}$, $P_{oc}$**




Rated voltage is applied to the transformer under no-load and exciting current flows, which is a few percent of rated current.

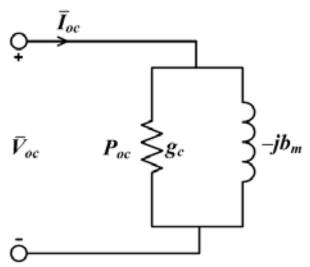



**where** $I_2' = 0$ **and** $I_e = I_1$



**Open circuit equivalent circuit**

# Open-circuit Test




$$g_c = \frac{P_{oc}}{V_{oc}^2}$$

$$|Y_c| = \frac{I_{oc}}{V_{oc}}$$

**where** $Y_c = g_c - j\,b_m$

$$b_m = \sqrt{|Y_c|^2 - g_c^2}$$

**where** $g_c = \dfrac{1}{r_c}$ **and** $b_m = \dfrac{1}{x_m}$

# 5. Voltage regulation (VR%)

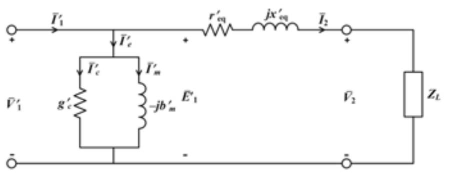
- The change in secondary terminal voltage (load voltage) from no-load to full-load
  - expressed as a percentage (%) of the rated value
  - ideally VR% = 0.

$$\mathbf{VR\%} = \frac{V_1 - V_2'}{V_{1\mathbf{(rated)}}} \times \mathbf{100\%}$$

or

$$\mathbf{VR\%} = \frac{V_1' - V_2}{V_{2\mathbf{(rated)}}} \times \mathbf{100\%}$$

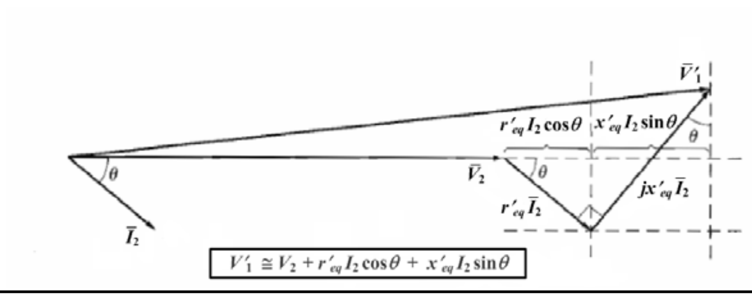
**Simplification by approximation**



$$\bar{V}_1' = \bar{V}_2 + \left(r_{eq}' + j\, x_{eq}'\right) \bar{\mathbf{I}}_2$$

$$V_1' \approx V_2 + I_2 r_{eq}' \cos\theta + I_2 x_{eq}' \sin\theta$$

$$\mathbf{VR\%} = \frac{I_2 r_{eq}' \cos\theta + I_2 x_{eq}' \sin\theta}{V_2} \times \mathbf{100\%}$$



## Zero regulation, i.e. VR% = 0

- For zero regulation, phase angle ($\theta$) of the load is given by

$$VR\% = \frac{I_2}{V_2}\left(r'_{eq}\cos\theta + x'_{eq}\sin\theta\right)\times 100\%$$

$$r'_{eq}\cos\theta + x'_{eq}\sin\theta = 0$$

$$\boxed{\theta = -\tan^{-1}\left(\frac{r'_{eq}}{x'_{eq}}\right)}$$

Note that $\theta < 0$, so

**Load must be capacitive for zero regulation**

## NOTE

- For an inductive load, $Z_L = R_L + jX_L$
  - Always $V'_1 > V_2$, i.e. **VR% > 0**
- For a capactive load, $Z_L = R_L - jX_L$
  - Usually $V'_1 \le V_2$, i.e. **VR% ≤ 0** (where $-x'_{eq}\sin\theta \ge r'_{eq}\cos\theta$)

# 6. Efficiency (η%)

- The ratio of the output power given to the load and the input power taken from the electrical supply
  - expressed as a percentage (%)

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} 100\%$$

$$\eta = \frac{P_{\text{out}}}{P_{\text{out}} + P_{\text{losses}}} 100\%$$

$$\eta = \frac{P_{\text{out}}}{P_{\text{out}} + P_{\text{losses}}} 100\%$$

$$P_{\text{losses}} = P_{\text{cu}} + P_{\text{core}}$$

$$\eta = \frac{P_{\text{out}}}{P_{\text{out}} + P_{\text{cu}_1} + P_{\text{cu}_2} + P_{\text{core}}} 100\%$$

$$P_{\text{cu}} = P_{\text{cu}_1} + P_{\text{cu}_2}$$

$$\eta = \frac{V_2 I_2 \cos\theta}{V_2 I_2 \cos\theta + I_1^2 r_1 + I_2^2 r_2 + V_1^2 g_c} 100\%$$

$$P_{\text{cu}} \approx I_2^2 r'_{eq}$$

$$\eta = \frac{V_2 I_2 \cos\theta}{V_2 I_2 \cos\theta + I_2^2 r'_{eq} + V_1^2 g_c} 100\%$$

or

$$\eta = \frac{V'_2 I'_2 \cos\theta}{V'_2 I'_2 \cos\theta + I'^2_2 r_{eq} + V_1^2 g_c} 100\%$$

# Maximum efficiency

Let us find the value of $I_2$ which maximizes the efficiency $\frac{d\eta}{dI_2} = 0$

where $\eta = \frac{V_2 I_2 \cos\theta}{V_2 I_2 \cos\theta + I_2^2 r'_{eq} + P_{\text{core}}}\ 100\%$

$$\frac{d\eta}{dI_2} = 0 \quad \Rightarrow \quad V_2 \cos\theta \left(V_2 I_2 \cos\theta + I_2^2 r'_{eq} + P_{\text{core}}\right) - \left(V_2 \cos\theta + 2 I_2 r'_{eq}\right) V_2 I_2 \cos\theta = 0$$

$$V_2 I_2 \cos\theta + I_2^2 r'_{eq} + P_{\text{core}} - \left(V_2 \cos\theta + 2 I_2 r'_{eq}\right) I_2 = 0$$

$$P_{\text{core}} - I_2^2 r'_{eq} = 0$$

$$P_{\text{core}} = I_2^2 r'_{eq}$$

$$\boxed{P_{\text{core}} = P_{\text{cu}}} \qquad \text{i.e.,} \quad V_1^2 g_c = I_2^2 r'_{eq} = I_2'^2 r_{eq}$$

Thus, maximum efficiency is achieved if core loss equals to the copper loss.

# Examples

1. A 12kVA, 220/440V, 50 Hz single phase transformer has the following test data:

   - No-load test: 220V, 2A, 165W (measured at primary side)
   - Short-circuit test: 12V, 15A, 60W (measured at secondary side)

a) Calculate the equivalent circuit parameters referred to primary side

b) Calculate the primary terminal voltage on full-load at a power factor of: 0.8 pf lagging.

2. Given a 250kVA, 4160:480V, 60 Hz transformer, the following parameters are obtained by tests
   - $r_1 = 0.09\ \Omega$ and $x_1 = 1.7\ \Omega$
   - $r_2 = 1.2 \times 10^{-3}\ \Omega$ and $x_2 = 2.26 \times 10^{-2}\ \Omega$

   Neglecting core losses,

a) Calculate the primary voltage and voltage regulation for rated load at 76% pf lagging
b) Repeat a) for a load of 76% pf leading
c) Calculate the transformer efficiency for parts a) and b) with a core loss $P_{core}$ = 547W.

3. The parameters of the exact equivalent circuit of a 150kVA, 2400/240V transformer are
   - $r_1 = r'_2 = 0.2\ \Omega$
   - $x_1 = x'_2 = 0.45\ \Omega$
   - $r_c = 10\ k\Omega$ and $x_m = 1.55\ k\Omega$

   Using both the exact and approximate equivalent circuit of the transformer, determine

   a) Voltage regulation
   b) Efficiency

   for rated load at 0.8pf lagging

4. At  10kVA, 8000:230V transformer has a leakage impedance referred to primary of 90+ j400Ω. Exciting branch parameters are
   - $r_c$ = 500 kΩ and $x_m$ = 60 kΩ

a) If primary voltage $V_1$ = 7967V and actual load impedance $Z_L$ = 4.2+3.15Ω, find the secondary voltage of the transformer
b) If the load is disconnected, and a capacitor of –j6Ω is connected in its place, what will be the load voltage?